Non-perturbative topological string theory on compact Calabi-Yau manifolds

w/ Jie Gu, Albrecht Klemm,

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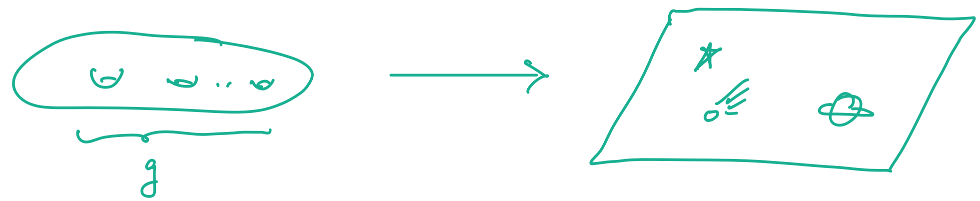
One slide summary of talk

Topological string theory starts out life as a worldsheet theory, just like string theory proper



$$F^{(0)}_{top} = \sum_{g=0}^{\infty} F_g \, g_s^{2g-2}$$

- intrinsically perturbative definition of F_{top}
- F_g grow factorially $\rightarrow$ F_{top} diverge factorially

Without providing non-perturbative definition of theory, we will compute correction to $F^{(0)}_{top}$ of the form

$$e^{-\ell A/g_s} \sum_{k=0}^{\infty} F_k^{(\ell)} \, g_s^{k-1}$$

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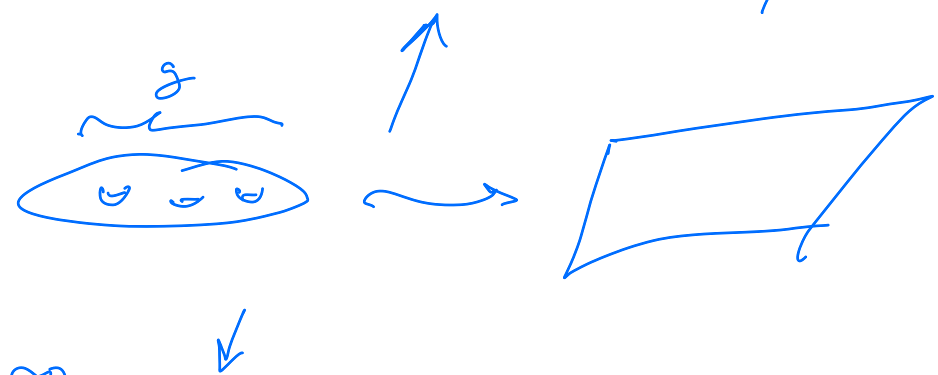

$$F = \sum_{g=0}^{\infty} F_g \, g_s^{2g-2}$$

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- F_g grow factorially $\rightarrow$ F diverge factorially

Without providing a non-perturbative definition of theory, we will compute corrections $e^{-\ell A/g_s} \sum_{k=0}^{\infty} F_k^{(\ell)} g_s^{k-1}$ exactly.

Structure of talk

1. Review of topological strings
2. Review of resurgence
3. Computing instanton corrections to F_{top}
4. Experimental evidence

1. Review of topological strings

Review of topological strings

Type II strings on $\mathbb{R}^{1,3} \times$ Calabi-Yau

Review of topological strings

Type II strings on $\mathbb{R}^{1,3} \times$ Calabi-Yau

distinguished class of
3 cplx dim'l mflds M

Review of topological strings

Type II strings on $\mathbb{R}^{1,3} \times$ Calabi-Yau

$\Rightarrow$ 4d theory with $\mathcal{N}=2$ supersymmetry

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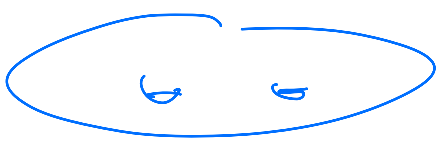
Worldsheet theory $\supset$ 2d $\mathcal{N}=(2,2)$ theory

 $\longrightarrow$ CY

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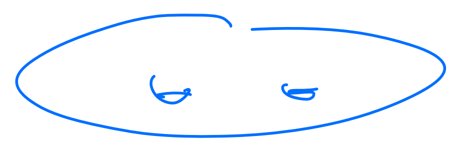 $\longrightarrow$ CY

$\big\downarrow$ twisting

topological string theory

Review of topological strings

Observables of topological string theory:

partition $f^{\underline{cn}}$ $\qquad F_g$

$\qquad\qquad\qquad\qquad\uparrow$

$\qquad\qquad\qquad$ genus of worldsheet

Review of topological strings

Observables of topological string theory:

partition $f^{\underline{cn}}$ $\quad F_g$

$\uparrow$

genus of worldsheet

F_g does not depend on moduli of Riemann surface

$\rightarrow$ the moduli space M_g is integrated over.

The B-model

The $\mathcal{F}_g$ are f^{cns}s on the complex structure moduli space $\mathcal{M}_{cplx}$ of M.

Eg. $X = \{ p(x_1, \ldots, x_5) = 0 \} \subset \mathbb{P}^4$

$\uparrow$ polynomial : coefficients z_i determine cplx structure

$\longrightarrow \quad \mathcal{F}_g(z_i)$

Why is the topological string interesting?

- Computable subsector of string theory

- Computes terms in effective 4d action of Calabi - Yau compactifications

- Counts BPS particles

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  ↳ enumerative geometry

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Distinguished points on moduli space

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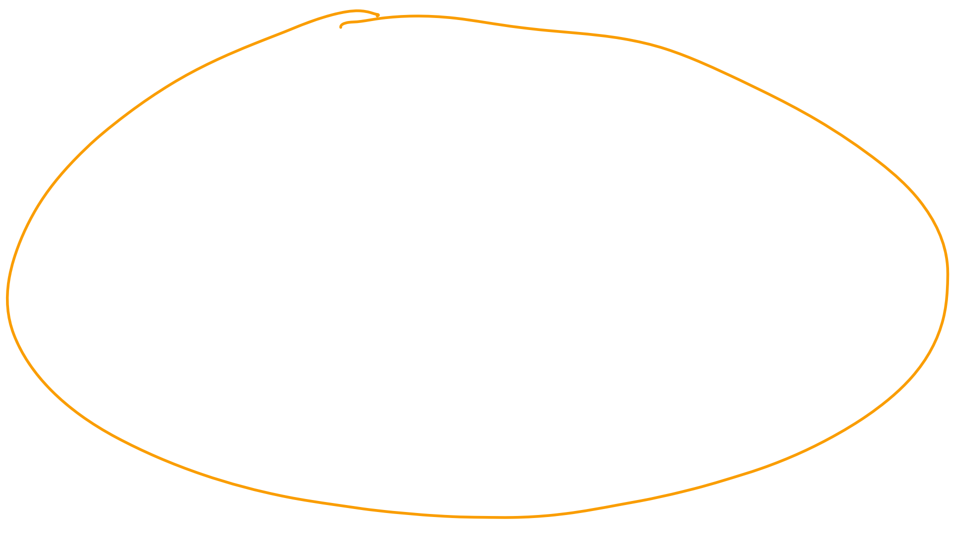

Mfld of one-parameter models

Distinguished points on moduli space

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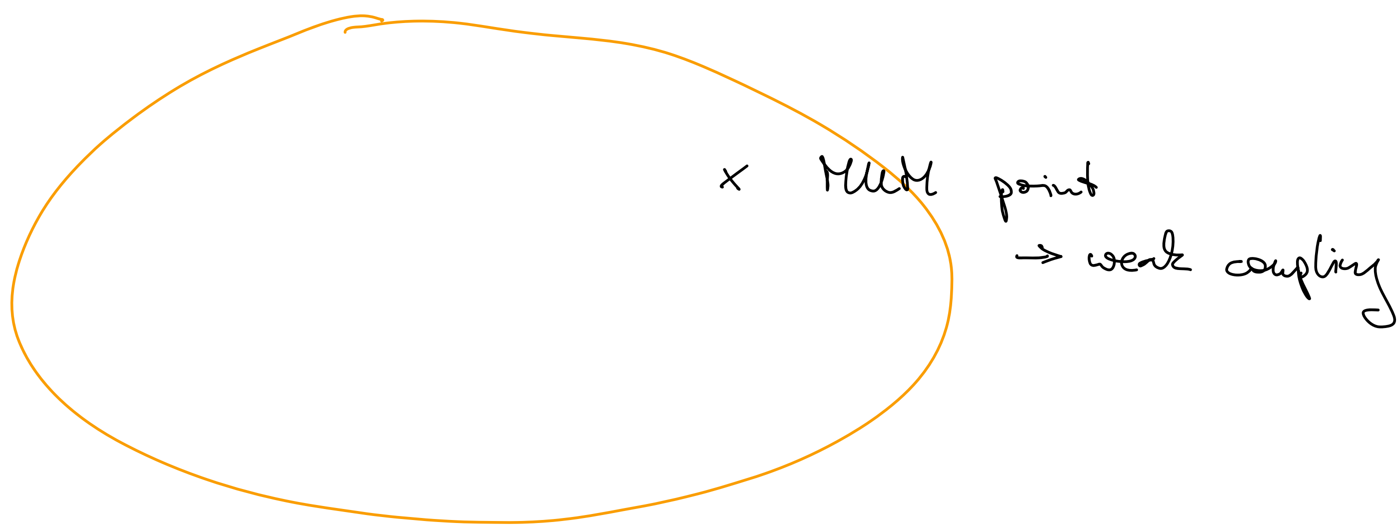

Mdle of one-parameter models

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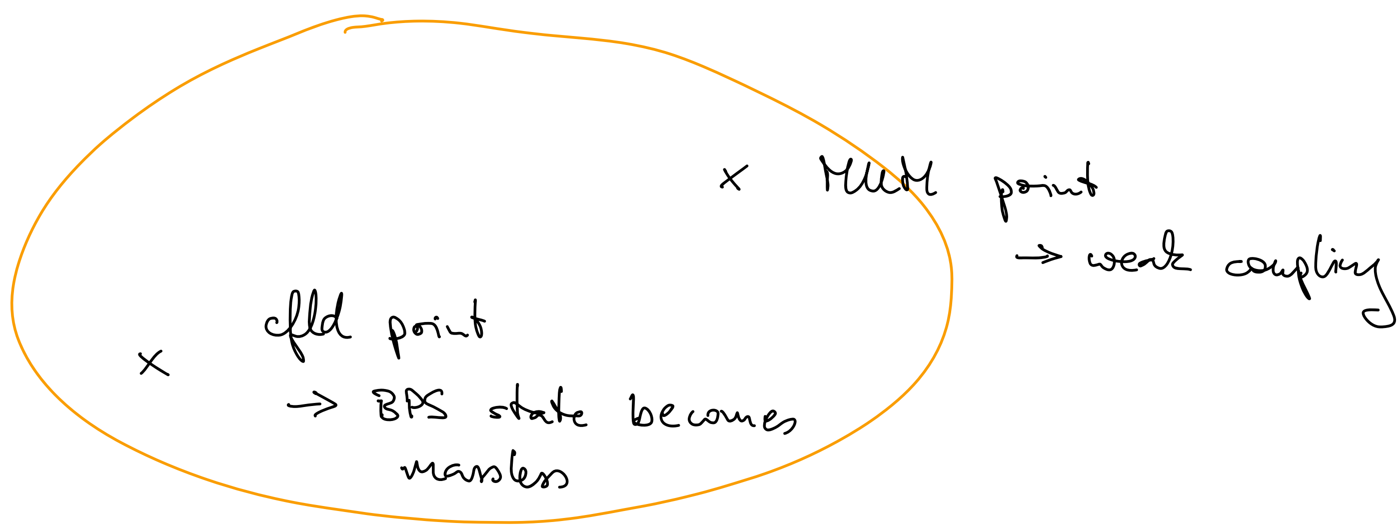

$\mathcal{M}_{cplx}$ of one-parameter models

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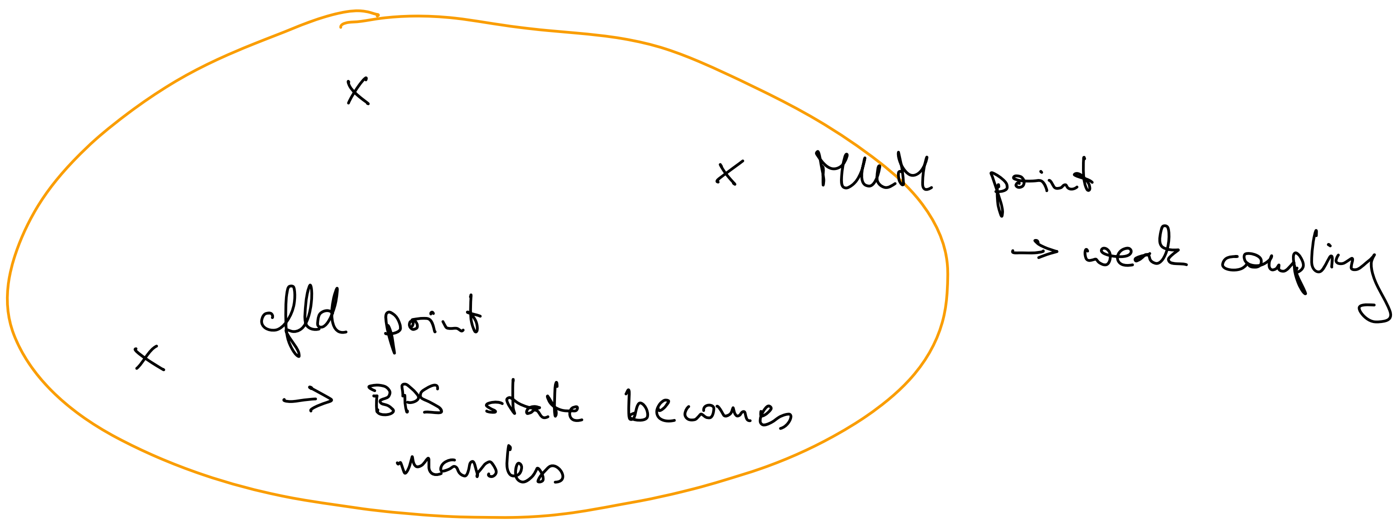

$\mathcal{M}_{cplx}$ of one-parameter models

Special geometry I

Extracting physics/enumerative information from F_g requires expressing them in terms of special coordinates:

by CY property $H^{3,0}(M) = \langle \Omega \rangle$

locally $\Omega = \Omega_{ijk}\, x^i x^j x^k$

Let $\{A_0, \dots, A_n, B^0, \dots, B^n\}$ be a basis of $H_3(M,\mathbb{Z})$

$$\Pi = \begin{pmatrix} \int_{B^0} \Omega \\ \vdots \\ \int_{B^n} \Omega \\ \int_{A_0} \Omega \\ \vdots \\ \int_{A_n} \Omega \end{pmatrix} =: \begin{pmatrix} P_0 \\ \vdots \\ P_n \\ X^0 \\ \vdots \\ X^n \end{pmatrix}$$

period vector

Special geometry I

Computable as power series (+ logarithms) solutions of differential equations anywhere on moduli space.

$$\Pi = \begin{pmatrix} \int_{B^0} \Omega \\ \vdots \\ \int_{B^n} \Omega \\ \int_{A_0} \Omega \\ \vdots \\ \int_{A_n} \Omega \end{pmatrix} =: \begin{pmatrix} P_0 \\ \vdots \\ P_n \\ X^0 \\ \vdots \\ X^n \end{pmatrix}$$

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Special geometry II

$$\Pi = \begin{pmatrix} \int_{B^0} \Omega \\ \vdots \\ \int_{B^n} \Omega \\ \int_{A_0} \Omega \\ \vdots \\ \int_{A_n} \Omega \end{pmatrix} =: \begin{pmatrix} P_0 \\ \vdots \\ P_n \\ X^0 \\ \vdots \\ X^n \end{pmatrix}$$ period vector

Locally, the X^I furnish projective coordinates on $\mathcal{M}_{cplx}$

$$z \longmapsto \left(X^0(z) : \ldots : X^n(z) \right) \in \mathbb{P}^{n+1}$$

Affine coordinates $\quad t^i(z) := X^i(z) / X^0(z)$

Special geometry III

A distinguished metric on $\mathcal{M}_{cplx}$ is induced by the Kähler form K defined via

$$e^{-K} = i\int \Omega \wedge \bar{\Omega}$$

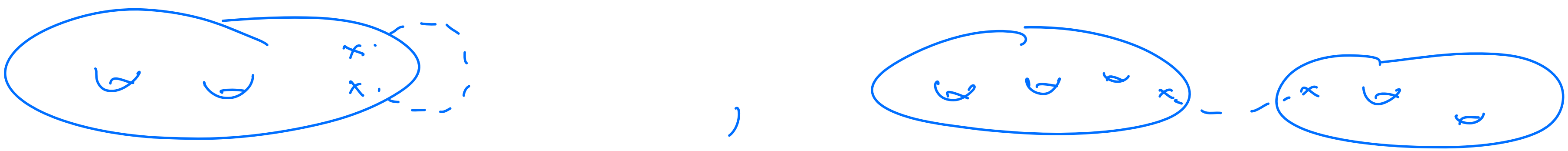
Note that K is <u>not</u> a holomorphic f^{cs} of z.

The holomorphic anomaly equations I

- The twisting procedure renders anti-holomorphic dependence of F_g Q-exact

  $\rightarrow$ F_g picks up $\bar{z}$ dependence only from $\partial \mathcal{M}_g$

$\Rightarrow$ sources of $\bar{z}$ dependence



The holomorphic anomaly equations I

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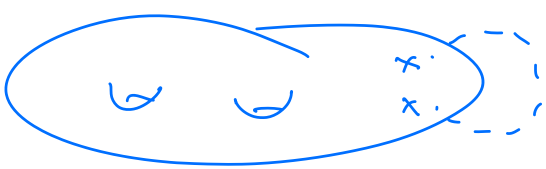

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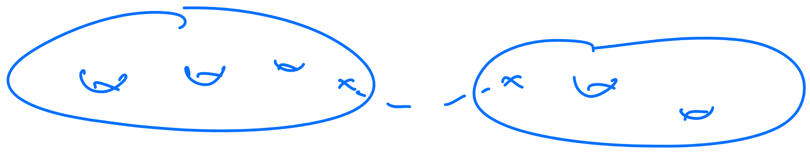
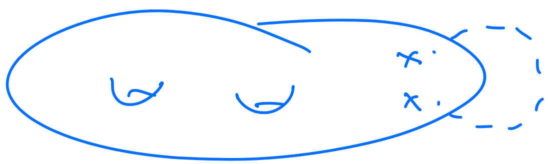


The holomorphic anomaly equations II

→ sources of $\bar z$ dependence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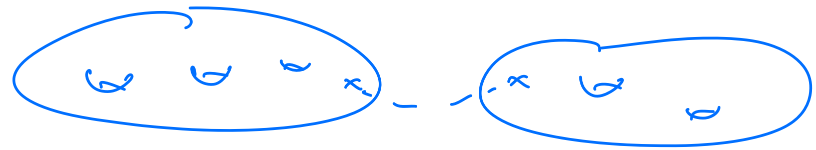
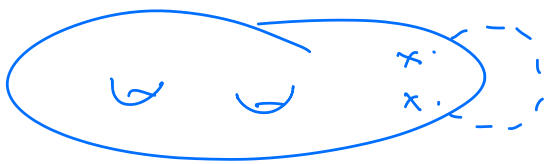


$$\partial_{\bar z} F_g = \frac{1}{2} C^{zz}_{\bar z} \left(D_z D_z F_{g-1} + \sum_{h=1}^{g-1} D_z F_{g-h} D_z F_h \right)$$

for $g \geq 2$

The holomorphic anomaly equations II

$\rightarrow$ sources of $\bar{z}$ dependence



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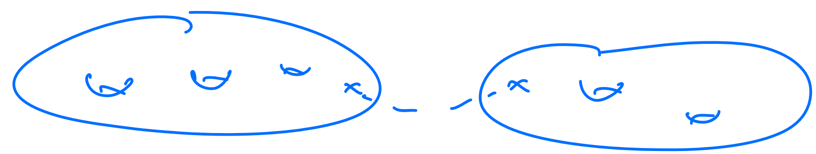

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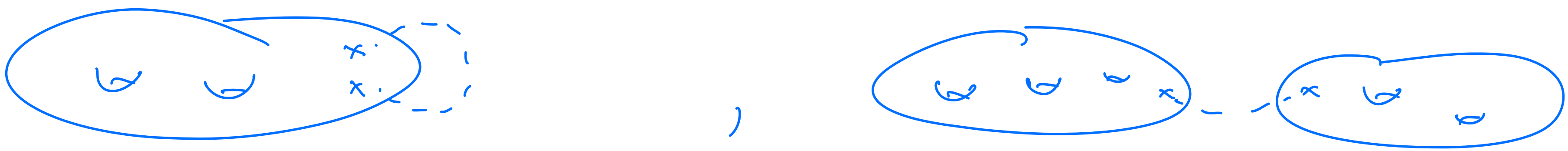
special geometry data determined $\sim$ terms of periods

The holomorphic anomaly equations II

$\Rightarrow$ sources of $\bar{z}$ dependence



$$\partial_{\bar{z}} \bar{F}_g = \frac{1}{2} \bar{C}_{\bar{z}}^{zz} \left(D_z D_z \bar{F}_{g-1} + \sum_{h=1}^{g-1} D_z \bar{F}_{g-h} D_z \bar{F}_h \right)$$

for $g \geq 2$

$\Rightarrow$ recursion relation for $\bar{F}_g$

up to purely holomorphic piece : $\bar{F}_g + f_g(z)$

The holomorphic anomaly equations III

The HAE imply the following structure theorem for the F_g: all non-holomorphic dependence is captured by a finite set of generators:

$$\{ \underbrace{S^{ik}, \quad S^k, \quad S}_{\text{propagators}}, \quad K \}$$

where $\quad \partial_{\bar{i}} S^{jk} = C_{\bar{i}}^{jk}, \quad \partial_{\bar{j}} S^k = G_{i\bar{j}} S^{ik}$

$$\partial_{\bar{j}} S = G_{i\bar{j}} S^i$$

The holomorphic anomaly equations IV

$$\partial_{\bar{i}} F_g = \frac{1}{2} C^{jk}_{\bar{i}} \left(D_j \partial_k F_{g-1} + \sum_{h=1}^{g-1} D_j F_{g-h} \partial_k F_h \right)$$

becomes

$$\frac{\partial F_g}{\partial S^{ij}} = \frac{1}{2} D_i D_j F_{g-1} + \frac{1}{2} \sum_{h=1}^{g-1} D_i F_h D_j F_{g-h} \ , \qquad \frac{\partial F_g}{\partial K_i} + S^i \frac{\partial F_g}{\partial S} + S^{ij} \frac{\partial F_g}{\partial S^j} = 0 \ .$$

Explicit form of F_2 for 1-parameter model

$$F_2 = \frac{5}{24} C_{111}^2 \left(S^{11}\right)^3 + \frac{1}{8}\left(\partial_1 C_{111} - 3 C_{111} q_{11}^1 + 4 C_{111} f_1^{(1)}\right)\left(S^{11}\right)^2$$
$$+ \left(\frac{1}{4} q_1^{11} C_{111} + \frac{1}{2}\partial_1 f_1^{(1)} + \frac{1}{2} f_1^{(1)}\left(f_1^{(1)} - q_{11}^1\right) + \frac{1}{2}\left(1 - \frac{\chi}{24}\right) q_{11}\right) S^{11}$$
$$+ \frac{\chi}{48}\left(C_{111} S^{11} + 2 f_1^{(1)}\right)\tilde{S}^1 + \frac{\chi}{24}\left(\frac{\chi}{24} - 1\right)\tilde{S} + f_2(z).$$

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Euler characteristic of $\tilde{M}$ (the mirror to M)

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special geometry data

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Propagators: anholomorphic objects

(K_i dependence absorbed in $\tilde{S}$, $\tilde{S}^1$)

Explicit form of F_2 for 1-parameter model

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holomorphic ambiguities of propagators

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This expression is valid everywhere on $\mathcal{M}_{cplx}$!

In a moment, we will rewrite the holomorphic anomaly equation in terms of $\mathcal{F}^{(0)} = \sum \mathcal{F}_g \, g_s^{2g-2}$,

but first ...

Back to physics: the holomorphic limit I

To describe physics in vicinity of $z^* \in \mathcal{M}_{cpl}$ requires <u>choice of frame</u>

1. Adapted choice of A-periods X^I to yield local coordinates

$$z_i \longrightarrow X^i / X^0$$

2. Specialization

$$\bar{z} \longrightarrow \overline{z^*}$$

holomorphic limit

Back to physics: the holomorphic limit I

2. Specialization $\quad \overline{Z} \xrightarrow[\text{holomorphic limit}]{} \overline{Z^*}$

Back to physics: the holomorphic limit II

2. Specialization $\quad z \xrightarrow{} \overline{z^*}$

holomorphic limit

In practice,

$$\mathcal{S}^{zz} = -\frac{1}{C}\left(\frac{w_{0,2}}{w_{0,1}} - q^z_{zz}\right), \qquad \tilde{\mathcal{S}}^z = -\frac{1}{C}\left(\frac{w_{1,2}}{w_{0,1}} - q_{zz}\right),$$

$$\tilde{\mathcal{S}} = -\frac{1}{2C}\left(\frac{w_{1,3}}{w_{0,1}} - \partial_z q_{zz} - q_{zz}q^z_{zz}\right) + \frac{\partial_z C}{2C^2}\left(\frac{w_{1,2}}{w_{0,1}} - q_{zz}\right) - \frac{q^z_z}{2} \,.$$

where $\quad w_{i,j} = \partial^j_z X^0_* \partial^i_z X^1_* - \partial^i_z X^0_* \partial^j_z X^1_*,$

Back to physics: the holomorphic limit III

$$F_g \longrightarrow \mathcal{F}_g$$

Fact: $\mathcal{F}_g(X^0, X^1, \ldots, X^n)$ is homogeneous of degree $2-2g$, i.e.

$$\mathcal{F}_g(X^0, X^1, \ldots, X^n) = (X^0)^{2-2g}\,\mathcal{F}_g(1, X^1/X^0, \ldots, X^n/X^0)$$

Getting our hands dirty in the MUM frame I

All genus structural results exist in MUM frame close to MUM point.

$$\mathcal{F}^{(0)}(X) = \frac{c(t)}{\lambda^2} + l(t) + \sum_{g\geq 0} \sum_{\beta\in H_2(\widetilde{M},\mathbb{Z})} \sum_{k\geq 1} \frac{n_{g,\beta}}{k}\left(2\sin\frac{k\lambda}{2}\right)^{2g-2} Q_\beta^k$$

↑ mirror to M

Gopakumar–Vafa form of

$$\mathcal{F}^{(0)}(X) = \sum_{g=0}^{\infty} \mathcal{F}_g(X^0,\dots,X^n)\, g_s^{2g-2}$$

Getting our hands dirty in the MaM frame II

special to
genus 0 and 1

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Getting our hands dirty in the MCM frame II

special to genus 0 and 1

GV invariants
→ integer enumerative invariants associated to curve class β

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Getting our hands dirty in the HCM frame II

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$e^{i\beta_i t^i}$

X^i/X^0

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$$\lambda = \frac{(2\pi i)^{3/2} g_s}{X^0}$$

$e^{i\beta_i t^i}$

X^i/X^0

Getting our hands dirty in the HCM frame II

$$\mathcal{F}^{(0)}(X) = \frac{c(t)}{\lambda^2} + l(t) + \sum_{g\geq 0}\sum_{\beta\in H_2(\widetilde{M},\mathbb{Z})}\sum_{k\geq 1}\frac{n_{g,\beta}}{k}\left(2\sin\frac{k\lambda}{2}\right)^{2g-2} Q_\beta^k$$

Getting our hands dirty in the HCM frame III

$$\mathcal{F}^{(0)}(X) = \frac{c(t)}{\lambda^2} + l(t) + \sum_{g\geq 0} \sum_{\beta\in H_2(\tilde{M},\mathbb{Z})} \sum_{k\geq 1} \frac{n_{g,\beta}}{k} \left(2\sin\frac{k\lambda}{2}\right)^{2g-2} Q_\beta^k$$

Asymptotics of $\mathcal{F}_g$ by expanding sin.

Getting our hands dirty in the MUM frame III

$$\mathcal{F}^{(0)}(X) = \frac{c(t)}{\lambda^2} + l(t) + \sum_{g\geq 0}\sum_{\beta\in H_2(\tilde{M},\mathbb{Z})}\sum_{k\geq 1}\frac{n_{g,\beta}}{k}\left(2\sin\frac{k\lambda}{2}\right)^{2g-2} Q_\beta^k$$

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Getting our hands dirty in the HCM frame III

$$\mathcal{F}^{(0)}(X) = \frac{c(t)}{\lambda^2} + l(t) + \sum_{g\geq 0}\sum_{\beta\in H_2(\widetilde{M},\mathbb{Z})}\sum_{k\geq 1}\frac{n_{g,\beta}}{k}\left(2\sin\frac{k\lambda}{2}\right)^{2g-2} Q_\beta^k$$

Asymptotics of $\mathcal{F}_g$ by expanding sin.

$$\sin^{-2}x = \sum_{n=0}^{\infty}(-1)^{n-1}\frac{4(2n-1)B_{2n}(2x)^{2n-2}}{(2n)!}\,, \quad \sin^2 x = \sum_{n=1}^{\infty}(-1)^{n-1}\frac{(2x)^{2n}}{2(2n)!}$$

Getting our hands dirty in the MCM frame III

$$\mathcal{F}^{(0)}(X) = \frac{c(t)}{\lambda^2} + l(t) + \sum_{g\geq 0} \sum_{\beta \in H_2(\tilde{M},\mathbb{Z})} \sum_{k\geq 1} \frac{n_{g,\beta}}{k} \left(2\sin\frac{k\lambda}{2}\right)^{2g-2} Q_\beta^k$$

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Bernoulli numbers $\quad B_{2n} = (2n)! \dfrac{2(-1)^{n-1}}{(2\pi)^{2n}} \zeta(2n)$

Getting our hands dirty in the MaM frame III

$$\mathcal{F}^{(0)}(X) = \frac{c(t)}{\lambda^2} + l(t) + \sum_{g\geq 0} \sum_{\beta\in H_2(\widetilde{M},\mathbb{Z})} \sum_{k\geq 1} \frac{n_{g,\beta}}{k}\left(2\sin\frac{k\lambda}{2}\right)^{2g-2} Q_\beta^k$$

Asymptotics of $\mathcal{F}_g$ by expanding sin.

$$\sin^{-2} x = \sum_{n=0}^{\infty}(-1)^{n-1}\frac{4(2n-1)B_{2n}(2x)^{2n-2}}{(2n)!}, \quad \sin^2 x = \sum_{n=1}^{\infty}(-1)^{n-1}\frac{(2x)^{2n}}{2(2n)!}$$

Bernoulli numbers $B_{2n} = (2n)!\,\frac{2(-1)^{n-1}}{(2\pi)^{2n}}\,\zeta(2n)$

Getting our hands dirty in the HLM frame III

$$\mathcal{F}^{(0)}(X) = \frac{c(t)}{\lambda^2} + l(t) + \sum_{g\geq 0}\sum_{\beta\in H_2(\widetilde{M},\mathbb{Z})}\sum_{k\geq 1}\frac{n_{g,\beta}}{k}\left(2\sin\frac{k\lambda}{2}\right)^{2g-2} Q_\beta^k$$

Asymptotics of $\mathcal{F}_g$ by expanding sin.

$$\sin^{-2}x = \sum_{n=0}^{\infty}(-1)^{n-1}\frac{4(2n-1)B_{2n}(2x)^{2n-2}}{(2n)!}, \quad \sin^2 x = \sum_{n=1}^{\infty}(-1)^{n-1}\frac{(2x)^{2n}}{2(2n)!}$$

Bernoulli numbers $B_{2n} = (2n)!\,\frac{2(-1)^{n-1}}{(2\pi)^{2n}}\,\zeta(2n)$

$$\left(\frac{\lambda}{g_s}\right)^{2-2g}\mathcal{F}_g(X) = \sum_{\beta\in H_2(M,\mathbb{Z})}\left((-1)^{g+1}\frac{(2g-1)B_{2g}}{(2g)!}n_{0,\beta} + \frac{2(-1)^g n_{2,\beta}}{(2g-2)!} + \cdots\right)\mathrm{Li}_{3-2g}(Q_\beta)$$

$$\mathrm{Li}_{3-2g}(Q_\beta) = \sum_{n=1}^{\infty}\frac{Q_\beta^n}{n^{3-2g}}$$

Getting our hands dirty in the MUM frame III

$$\mathcal{F}^{(0)}(X) = \frac{c(t)}{\lambda^2} + l(t) + \sum_{g\geq 0}\ \sum_{\beta\in H_2(\tilde{M},\mathbb{Z})}\ \sum_{k\geq 1} \frac{n_{g,\beta}}{k}\left(2\sin\frac{k\lambda}{2}\right)^{2g-2} Q_\beta^k$$

Asymptotics of $\mathcal{F}_g$ by expanding sin.

$$\sin^{-2} x = \sum_{n=0}^{\infty}(-1)^{n-1}\frac{4(2n-1)B_{2n}(2x)^{2n-2}}{(2n)!}, \quad \sin^2 x = \sum_{n=1}^{\infty}(-1)^{n-1}\frac{(2x)^{2n}}{2(2n)!}$$

Bernoulli numbers $\quad B_{2n} = (2n)!\,\dfrac{2(-1)^{n-1}}{(2\pi)^{2n}}\,\zeta(2n)$

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$$\mathrm{Li}_{3-2g}(Q_\beta) = \sum_{n=1}^{\infty}\frac{Q_\beta^n}{n^{3-2g}}$$

2nd source of factorials $\longrightarrow$ $\displaystyle\sum_{n=1}^{\infty}\frac{Q_\beta^n}{n^{3-2g}}$

Getting our hands dirty in the VCM frame III

$$\left(\frac{\lambda}{g_s}\right)^{2-2g}\mathcal{F}_g(X) = \sum_{\beta\in H_2(\widetilde{M},\mathbb{Z})}\left((-1)^{g+1}\frac{(2g-1)B_{2g}}{(2g)!}n_{0,\beta}+\frac{2(-1)^g n_{2,\beta}}{(2g-2)!}+\cdots\right)\mathrm{Li}_{3-2g}(Q_\beta)$$

Getting our hands dirty in the HCM frame IV

$$\left(\frac{\lambda}{g_s}\right)^{2-2g}\mathcal{F}_g(X) = \sum_{\beta\in H_2(\tilde{M},\mathbb{Z})}\left((-1)^{g+1}\frac{(2g-1)B_{2g}}{(2g)!}n_{0,\beta} + \frac{2(-1)^g n_{2,\beta}}{(2g-2)!} + \cdots\right)\mathrm{Li}_{3-2g}(Q_\beta)$$

Getting our hands dirty in the MCM frame IV

$$\left(\frac{\lambda}{g_s}\right)^{2-2g}\mathcal{F}_g(X)=\sum_{\beta\in H_2(\tilde{M},\mathbb{Z})}\left((-1)^{g+1}\frac{(2g-1)B_{2g}}{(2g)!}n_{0,\beta}+\frac{2(-1)^g n_{2,\beta}}{(2g-2)!}+\cdots\right)\mathrm{Li}_{3-2g}(Q_\beta)$$

Euler characteristic of M

At $\beta=0$, with $n_{0,0}=\chi/2$ and $\mathrm{Li}_{3-2g}(1)=\zeta(3-2g)$

$$=B_{2g-2}/2g-2$$

$$\mathcal{F}_g(X)\sim-\frac{\chi}{2\pi^2}\left(1+\frac{1}{2g-2}\right)(\mathcal{N}X^0)^{2-2g}\,\Gamma(2g-1)$$

normalization constant

We can get similar results for $\beta \neq 0$

$(\chi \to n_{0,\beta})$, and also

at conifold in conifold frame,

but let's get back to the holomorphic

anomaly!

The holomorphic anomaly equation for $F^{(0)}$

As F_0 and F_1 are special, introduce

$$\widetilde{F}^{(0)} = F^{(0)} - g_s^{-2} F_0 = \sum_{g\geq 1} F_g g_s^{2g-2}$$ and $$\widehat{F}^{(0)} = \widetilde{F}^{(0)} - F_1 = \sum_{g\geq 2} F_g g_s^{2g-2}$$

$\Longrightarrow$

$$\frac{\partial \widehat{F}^{(0)}}{\partial S^{ij}} - \frac{1}{2} K_i \frac{\partial \widehat{F}^{(0)}}{\partial \tilde{S}^j} - \frac{1}{2} K_j \frac{\partial \widehat{F}^{(0)}}{\partial \tilde{S}^i} + \frac{1}{2} \frac{\partial \widehat{F}^{(0)}}{\partial \tilde{S}} K_i K_j = \frac{g_s^2}{2} \hat{D}_i \hat{D}_j \widetilde{F}^{(0)} + \frac{g_s^2}{2} \hat{D}_i \widetilde{F}^{(0)} \hat{D}_j \widetilde{F}^{(0)} ,$$

$$\frac{\partial \widehat{F}^{(0)}}{\partial K_i} = 0 .$$

Idea :

Forget perturbative roots of this equation!

$$
\frac{\partial \widehat{F}^{(\cancel{0})}}{\partial S^{ij}} - \frac{1}{2} K_i \frac{\partial \widehat{F}^{(\cancel{0})}}{\partial \tilde{S}^j} - \frac{1}{2} K_j \frac{\partial \widehat{F}^{(\cancel{0})}}{\partial \tilde{S}^i} + \frac{1}{2} \frac{\partial \widehat{F}^{(\cancel{0})}}{\partial \tilde{S}} K_i K_j = \frac{g_s^2}{2} \hat{D}_i \hat{D}_j \widetilde{F}^{(\cancel{0})} + \frac{g_s^2}{2} \hat{D}_i \widetilde{F}^{(\cancel{0})} \hat{D}_j \widetilde{F}^{(\cancel{0})} \,,
$$

$$
\frac{\partial \widehat{F}^{(\cancel{0})}}{\partial K_i} = 0 \,.
$$

OK . . .

Now what ?

2. Review of resurgence

Resurgence

How to extract non-perturbative information from a formal power series?

Borel resummation I

Consider a factorially divergent series

$$\varphi(z) = \sum a_n z^n \quad , \quad a_n \sim n!$$

formal power series

Improve convergence by considering Borel transform

$$\hat{\varphi}(\xi) = \sum \frac{a_n}{n!} \xi^n$$

If $\hat{\varphi}(\xi)$ exists around the origin and can be analytically continued to the complex ξ-plane

Borel plane

$\rightarrow$ φ is a resurgent fct

Borel resummation II

Relevance of $\hat{\varphi}(\xi) = \sum \frac{a_n}{n!} \xi^n$?

Consider its Laplace transform

$$\int_0^\infty \hat{\varphi}(\xi z)\, e^{-\xi}\, d\xi$$

$$\longrightarrow \sum \int_0^\infty \frac{a_n}{n!} (\xi z)^n e^{-\xi} d\xi$$

$$= \sum a_n z^n \frac{1}{n!} \underbrace{\int_0^\infty \xi^n e^{-\xi} d\xi}_{= n!}$$

$$= \varphi(z)$$

Borel resummation II

φ Laplace transform exists,

$$\int_0^\infty \hat{\varphi}(\xi z)\, e^{-\xi}\, d\xi \quad \sim \quad \sum a_n z^n$$

$\uparrow$

asymptotic expansion

Terminology

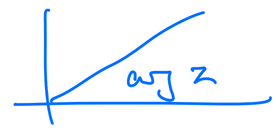
$$s(\varphi)(z) = \int_0^\infty \hat{\varphi}(\xi z)\, e^{-\xi}\, d\xi$$

is the Borel resummation of the formal power series φ.

What can go wrong?

$$s(\varphi)(z) = \int_0^\infty \hat{\varphi}(\xi z)\, e^{-\xi}\, d\xi$$

$$= \frac{1}{z} \int \hat{\varphi}(\xi)\, e^{-\xi/z}\, d\xi$$



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$$s(\varphi)(z) = \int_0^\infty \hat{\varphi}(\xi z)\, e^{-\xi}\, d\xi$$

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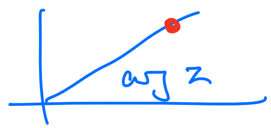
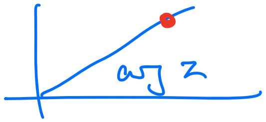


If $\hat{\varphi}(\xi)$ has a singularity on integration path,

What can go wrong?

$$s(\varphi)(z) = \int_0^\infty \hat{\varphi}(\xi z)\, e^{-\xi}\, d\xi$$

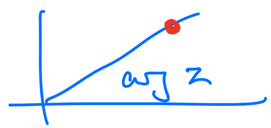
$$= \frac{1}{z} \int \hat{\varphi}(\xi)\, e^{-\xi/z}\, d\xi$$



If $\hat{\varphi}(\xi)$ has a singularity on integration path, integral is ill defined.

Lateral Borel resummation

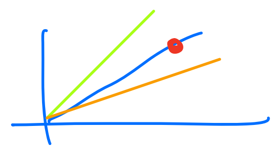
$$s(\varphi)(z) = \frac{1}{z} \int \hat{\varphi}(\xi)\, e^{-\xi/z}\, d\xi$$



If $\hat{\varphi}(\xi)$ has a singularity on integration path, integral is ill defined.

Lateral Borel resummation

$$s^{\pm}(\varphi)(z) = \frac{1}{z}\int \hat{\varphi}(\xi)\, e^{-\xi/z}\, d\xi$$



Integrate above or below singularity:

$$\left.\begin{array}{l} s^{+}(\varphi)(z) \\ s^{-}(\varphi)(z) \end{array}\right\} \sim \varphi(z)$$

$\Rightarrow$ $s^{+}(\varphi)(z) - s^{-}(\varphi)(z) \sim 0$ eg. $e^{-1/z}$

Singularities in the Borel plane I

Let Ω be indexing set of singular points S, i.e.

$$S = \{ \xi_\omega \mid \omega \in \Omega \}$$

Common case: logarithmic singularity at ξ_ω

Stokes constant

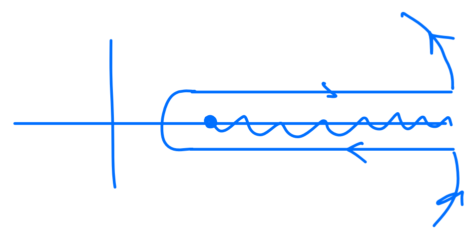
$$\hat{\varphi}(\xi_\omega + \xi) = -\frac{S_\omega}{2\pi} \log(\xi)\, \hat{\varphi}_\omega(\xi) + \text{regular}$$

$$\hat{\varphi}_\omega(\xi) = \sum_{n=0}^{\infty} \hat{c}_n \xi^n$$ has finite radius of convergence

Singularities in the Borel plane II

$$\hat{\varphi}(\xi_\omega + \xi) = -\frac{S_\omega}{2\pi}\log(\xi)\,\hat{\varphi}_\omega(\xi) + \text{regular}$$

$$s_+(\varphi)(z) - s_-(\varphi)(z) = \int \hat{\varphi}(\xi)\, e^{-\xi/z}\, d\xi$$



$$= i S_\omega\, e^{-\xi_\omega/z}\, s_-(\varphi_\omega)(z)$$

exponentially suppressed

Singularities in the Borel plane III

$$s_+(\varphi)(z) - s_-(\varphi)(z) = i S_\omega e^{-\xi\omega/z} s_-(\varphi_\omega)(z)$$

$$\Rightarrow \quad s_+(\varphi)(z) = s_-\big(\underbrace{\varphi + i S_\omega e^{-\xi\omega/z} \varphi_\omega}_{=: \mathfrak{S}_\omega(\varphi)}\big)(z)$$

$\uparrow$ Stokes automorphism

$\mathfrak{S}_\omega(\varphi)$ is a trans-series:

a generalization of the notion of formal power series, analytically controlled upon Borel resummation

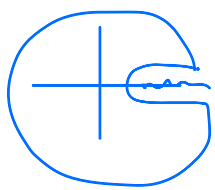
Asymptotics of perturbative coefficients

$$\hat{\varphi}(\zeta) = \sum \frac{a_n}{n!} \zeta^n$$

knows about all perturbative coefficients a_n

$$\frac{a_n}{n!} = \frac{1}{2\pi i} \oint \frac{\hat{\varphi}(\zeta)}{\zeta^{n+1}} \, d\zeta$$

$$= \frac{1}{2\pi i} \int \frac{\hat{\varphi}(\zeta)}{\zeta^{n+1}} \, d\zeta$$



$$\sim \frac{1}{n!} \frac{S_\omega}{2\pi} \sum_{k=0}^{\infty} c_k (\zeta_\omega)^{k-n} \, \Gamma(n-k)$$

$n \gg 1$

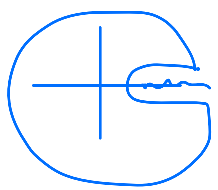
Comparing to asymptotics of F_g

$$\hat{\varphi}(\xi) = \sum \frac{a_n}{n!} \xi^n$$

knows about all perturbative coefficients a_n

$$\frac{a_n}{n!} = \frac{1}{2\pi i} \oint \frac{\hat{\varphi}(\xi)}{\xi^{n+1}} d\xi$$

$$= \frac{1}{2\pi i} \int \frac{\hat{\varphi}(\xi)}{\xi^{n+1}} d\xi$$



$$\underset{n \gg 1}{\sim} \frac{1}{n!} \frac{S_\omega}{2\pi} \sum_{k=0}^{\infty} c_k (\xi_\omega)^{k-n} \Gamma(n-k)$$

Comparing to asymptotics of F_g

$$\frac{a_n}{n!} =$$

$$\underset{n \gg 1}{\sim} \frac{1}{n!} \frac{S_\omega}{2\pi} \sum_{k=0}^{\infty} c_k (\xi_\omega)^{k-n} \Gamma(n-k)$$

Comparing to asymptotics of $\mathcal{F}_g$

$$a_n \sim \frac{S_\omega}{2\pi} \sum_{k=0}^{\infty} c_k \left(\xi_\omega\right)^{k-n} \Gamma(n-k)$$

Comparing to asymptotics of $\mathcal{F}_g$

$$a_n \sim \frac{S_\omega}{2\pi} \sum_{k=0}^{\infty} c_k (\xi_\omega)^{k-n} \Gamma(n-k)$$

$$\mathcal{F}_g(X) \sim -\frac{\chi}{2\pi^2}\left(1+\frac{1}{2g-2}\right)(N X^0)^{2-2g}\,\Gamma(2g-1)$$

Comparing to asymptotics of $\mathcal{F}_g$

$$a_n \sim \frac{S_\omega}{2\pi} \sum_{k=0}^{\infty} c_k \left(\xi_\omega\right)^{k-n} \Gamma(n-k)$$

$$\mathcal{F}_g(X) \sim -\frac{\chi}{2\pi^2}\left(1+\frac{1}{2g-2}\right)\left(\mathcal{N} X^0\right)^{2-2g} \Gamma(2g-1)$$

$$= -\frac{\chi}{4\pi^2}\left(1+\frac{1}{2g-2}\right)\left[\left(\mathcal{N} X^0\right)^{2-2g} + \left(-\mathcal{N} X^0\right)^{2-2g}\right] \times \Gamma(2g-1)$$

$g_s \leftrightarrow -g_s$

symmetry

Comparing to asymptotics of $\mathcal{F}_g$

$$a_n \sim \frac{S_\omega}{2\pi} \sum_{k=0}^{\infty} c_k \left(\xi_\omega\right)^{k-n} \Gamma(n-k)$$

$$\mathcal{F}_g(X) \sim -\frac{X}{2\pi^2}\left(1 + \frac{1}{2g-2}\right)\left(\mathcal{N} X^0\right)^{2-2g} \Gamma(2g-1)$$

$$= -\frac{X}{4\pi^2}\left(1 + \frac{1}{2g-2}\right)\left[\left(\mathcal{N} X^0\right)^{2-2g} + \left(-\mathcal{N} X^0\right)^{2-2g}\right] \times \Gamma(2g-1)$$

$$= -\frac{X}{2\pi}\left[\frac{\mathcal{N} X^0}{2\pi}\left(\mathcal{N} X^0\right)^{1-2g} \Gamma(2g-1) + \frac{1}{2\pi}\left(\mathcal{N} X^0\right)^{2-2g} \Gamma(2g-2) + X^0 \to -X^0\right]$$

Comparing to asymptotics of $\mathcal{F}_g$

$$a_n \sim \frac{S_\omega}{2\pi} \sum_{k=0}^{\infty} c_k \left(\xi_\omega\right)^{k-n} \Gamma(n-k)$$

$$\mathcal{F}_g(X) \sim -\frac{X}{2\pi^2}\left(1+\frac{1}{2g-2}\right)\left(\mathcal{N} X^0\right)^{2-2g} \Gamma(2g-1)$$

$$= -\frac{X}{4\pi^2}\left(1+\frac{1}{2g-2}\right)\left[\left(\mathcal{N} X^0\right)^{2-2g} + \left(-\mathcal{N} X^0\right)^{2-2g}\right] \times \Gamma(2g-1)$$

$$= -\frac{X}{2\pi}\left[\frac{\mathcal{N} X^0}{2\pi}\left(\mathcal{N} X^0\right)^{1-2g} \Gamma(2g-1) + \frac{1}{2\pi}\left(\mathcal{N} X^0\right)^{2-2g} \Gamma(2g-2) + X^0 \to -X^0\right]$$

$$\Rightarrow \quad S_{\text{MuM}} = -X, \quad \xi_\omega = \mathcal{N} X^0,$$

$$c_0 = \frac{\xi_\omega}{2\pi}, \quad c_1 = \frac{1}{2\pi}, \quad c_{k\geq 2} = 0$$

Lessons

1. Stokes constant integer, related to geometric data

2. Singularities at integral periods

3. In this case, $c_0 = \frac{\mathcal{E}\omega}{2\pi}$, $c_1 = \frac{1}{2\pi}$,

$$c_{k\geq 2} = 0$$

OK . . .

Now what ?

OK . . .

Now what ?

→ Apply ideas from resurgence to F_{top},
in particular : make trans-series ansatz

Couso-Santamaría, Edelstein, Schiappa,
Vonk

3. Computing instanton corrections to $\overline{F}_{top}$

Solving HAE for trans-series ansatz I

Fact : singularities ξ_ω in Borel plane occur in integral multiples : $\underbrace{\xi_\omega ,\ 2\xi_\omega ,\ 3\xi_\omega}_{\xi_\omega \text{ instanton sector}}$

$$F = F^{(0)} + \sum_\omega F^{(\omega)}$$

leading singularity

$$= F^{(0)} + \sum_{\ell \geq 1} C^\ell F^{(\ell A)} + \ldots$$

book keeping device

For simplicity : $F^{(\ell A)} \longrightarrow F^{(\ell)}$

c_k above

$$F^{(1)} = e^{-A/g_s} \sum_{n \geq 0} F_n^{(1)} g_s^{n-1}$$

Solving HAE for trans-series ansatz II

$$F^{(1)} = e^{-\mathcal{A}/g_s} \sum_{n \geq 0} F_n^{(1)} g_s^{n-1}$$

Plug into HAE, solve for leading order in g_s:

$$F_0^{(1)} = f_0^{(1)} \exp\left(\frac{1}{2}\left(\partial_z \mathcal{A}\right)^2 S^{zz} - \mathcal{A}\partial_z\mathcal{A}\,\tilde{S}^z + \mathcal{A}^2\tilde{S}\right)$$

undetermined holomorphic f^{cn}

→ need boundary conditions!

Boundary conditions for non-holomorphic HAE I

Based on exact results at MUM and conifold pt:

<u>Conjecture</u>: Specialized to any frame in which $\mathcal{A}$ coincides with one of the A-periods,

$$F^{(1)} \longmapsto \mathcal{F}^{(1)}_{\mathcal{A}} = \frac{1}{2\pi}\left(\frac{\mathcal{A}}{g_s}+1\right)e^{-\mathcal{A}/g_s}$$

i.e. $\mathcal{F}^{(1)}_{0,\mathcal{A}} = \frac{1}{2\pi}\mathcal{A}$, $\quad \mathcal{F}^{(1)}_{1,\mathcal{A}} = \frac{1}{2\pi}$,

$$\mathcal{F}^{(1)}_{n,\mathcal{A}} = 0 \quad \text{for } n \geq 2$$

Boundary conditions for non-holomorphic HAE I

Conjecture :

$$F^{(1)}_{0,A} = \frac{1}{2\pi} A \,, \qquad F^{(1)}_{1,A} = \frac{1}{2\pi} \,,$$

$$F^{(1)}_{n,A} = 0 \qquad \text{for } n \geq 2$$

Boundary conditions for non-holomorphic HAE I

<u>Conjecture</u>: $\mathcal{F}^{(1)}_{0,A} = \frac{1}{2\pi} A$, $\mathcal{F}^{(1)}_{1,A} = \frac{1}{2\pi}$,

$$\mathcal{F}^{(1)}_{n,A} = 0 \quad \text{for } n \geq 2$$

Boundary conditions for non-holomorphic HAE II

<u>Conjecture</u>: $\mathcal{F}^{(1)}_{0,\mathcal{A}} = \frac{1}{2\pi}\mathcal{A}$, $\quad \mathcal{F}^{(1)}_{1,\mathcal{A}} = \frac{1}{2\pi}$,

$$\mathcal{F}^{(1)}_{n,\mathcal{A}} = 0 \quad \text{for } n \geq 2$$

$$F_0^{(1)} = f_0^{(1)} \exp\left(\frac{1}{2}\left(\partial_z \mathcal{A}\right)^2 S^{zz} - \mathcal{A}\partial_z\mathcal{A}\,\tilde{S}^z + \mathcal{A}^2\tilde{S}\right)$$

↑

undetermined holomorphic fcn

→ need boundary conditions!

Boundary conditions for non-holomorphic HAE II

Conjecture: $\mathcal{F}^{(1)}_{0,\mathcal{A}} = \frac{1}{2\pi}\mathcal{A}$, $\mathcal{F}^{(1)}_{1,\mathcal{A}} = \frac{1}{2\pi}$,

$$\mathcal{F}^{(1)}_{n,\mathcal{A}} = 0 \quad \text{for } n \geq 2$$

$$F_0^{(1)} = f_0^{(1)} \exp\left(\frac{1}{2}(\partial_z \mathcal{A})^2 S^{zz} - \mathcal{A}\partial_z\mathcal{A}\,\tilde{S}^z + \mathcal{A}^2\tilde{S}\right)$$

$$\rightarrow \quad \mathcal{F}^{(1)}_{0,\mathcal{A}} = f_0^{(1)} \exp\left(\frac{1}{2}(\partial_z \mathcal{A})^2 \mathcal{S}^{zz}_{\mathcal{A}} - \mathcal{A}\partial_z\mathcal{A}\,\tilde{\mathcal{S}}^z_{\mathcal{A}} + \mathcal{A}^2\tilde{\mathcal{S}}_{\mathcal{A}}\right)$$

Boundary conditions for non-holomorphic HAE II

Conjecture : $\mathcal{F}^{(1)}_{0,\mathcal{A}} = \frac{1}{2\pi}\mathcal{A}$, $\mathcal{F}^{(1)}_{1,\mathcal{A}} = \frac{1}{2\pi}$,

$$\mathcal{F}^{(1)}_{n,\mathcal{A}} = 0 \quad \text{for } n \geq 2$$

$$F_0^{(1)} = f_0^{(1)} \exp\left(\frac{1}{2}(\partial_z \mathcal{A})^2 \, S^{zz} - \mathcal{A}\partial_z\mathcal{A}\,\tilde{S}^z + \mathcal{A}^2\tilde{S}\right)$$

$$\rightarrow \quad \mathcal{F}^{(1)}_{0,\mathcal{A}} = f_0^{(1)} \exp\left(\frac{1}{2}(\partial_z \mathcal{A})^2 \, \mathcal{S}^{zz}_{\mathcal{A}} - \mathcal{A}\partial_z\mathcal{A}\,\tilde{\mathcal{S}}^z_{\mathcal{A}} + \mathcal{A}^2\tilde{\mathcal{S}}_{\mathcal{A}}\right)$$

$$\overset{!}{=} \frac{1}{2\pi}\mathcal{A}$$

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$$F_0^{(1)} = f_0^{(1)} \exp\left(\frac{1}{2}\left(\partial_z \mathcal{A}\right)^2 S^{zz} - \mathcal{A}\partial_z\mathcal{A}\,\tilde{S}^z + \mathcal{A}^2\tilde{S}\right)$$

$$\to \quad \mathcal{F}^{(1)}_{0,\mathcal{A}} = f_0^{(1)} \exp\left(\frac{1}{2}\left(\partial_z \mathcal{A}\right)^2 \mathcal{S}^{zz}_{\mathcal{A}} - \mathcal{A}\partial_z\mathcal{A}\,\tilde{\mathcal{S}}^z_{\mathcal{A}} + \mathcal{A}^2\tilde{\mathcal{S}}_{\mathcal{A}}\right)$$

$$\stackrel{!}{=} \frac{1}{2\pi}\mathcal{A}$$

$$\Rightarrow \quad F_0^{(1)} = \frac{1}{2\pi}\mathcal{A}\exp\left(\frac{1}{2}\left(\partial_z \mathcal{A}\right)^2\left(S^{zz} - \mathcal{S}^{zz}_{\mathcal{A}}\right) - \mathcal{A}\partial_z\mathcal{A}\left(\tilde{S}^z - \tilde{\mathcal{S}}^z_{\mathcal{A}}\right) + \mathcal{A}^2(\tilde{S} - \tilde{\mathcal{S}}_{\mathcal{A}})\right)$$

Valid in any frame!

All order solution for one-instanton sector

With much more work, we arrive at closed formula for $\mathcal{F}^{(1)}$.

To avoid introducing too much notation, let's specialize to a frame $\{X^I, P_I\}$ in the instanton sector

$$\mathcal{A} = c^I P_I + d_I X^I$$

$$\mathcal{F}^{(1)} = \frac{1}{2\pi}\left(1 + g_s c^J \frac{\partial \mathcal{F}}{\partial X^J}\left(X^I - g_s c^I\right)\right) \exp\left[\mathcal{F}\left(X^I - g_s c^I\right) - \mathcal{F}\left(X^I\right)\right].$$

$$\mathcal{F} = \frac{1}{g_s^2}\tilde{\mathcal{F}}_0 + \tilde{\mathcal{F}}_1 + \sum_{g\geq 2} g_s^{2g-2}\mathcal{F}_g$$

4. Experimental evidence

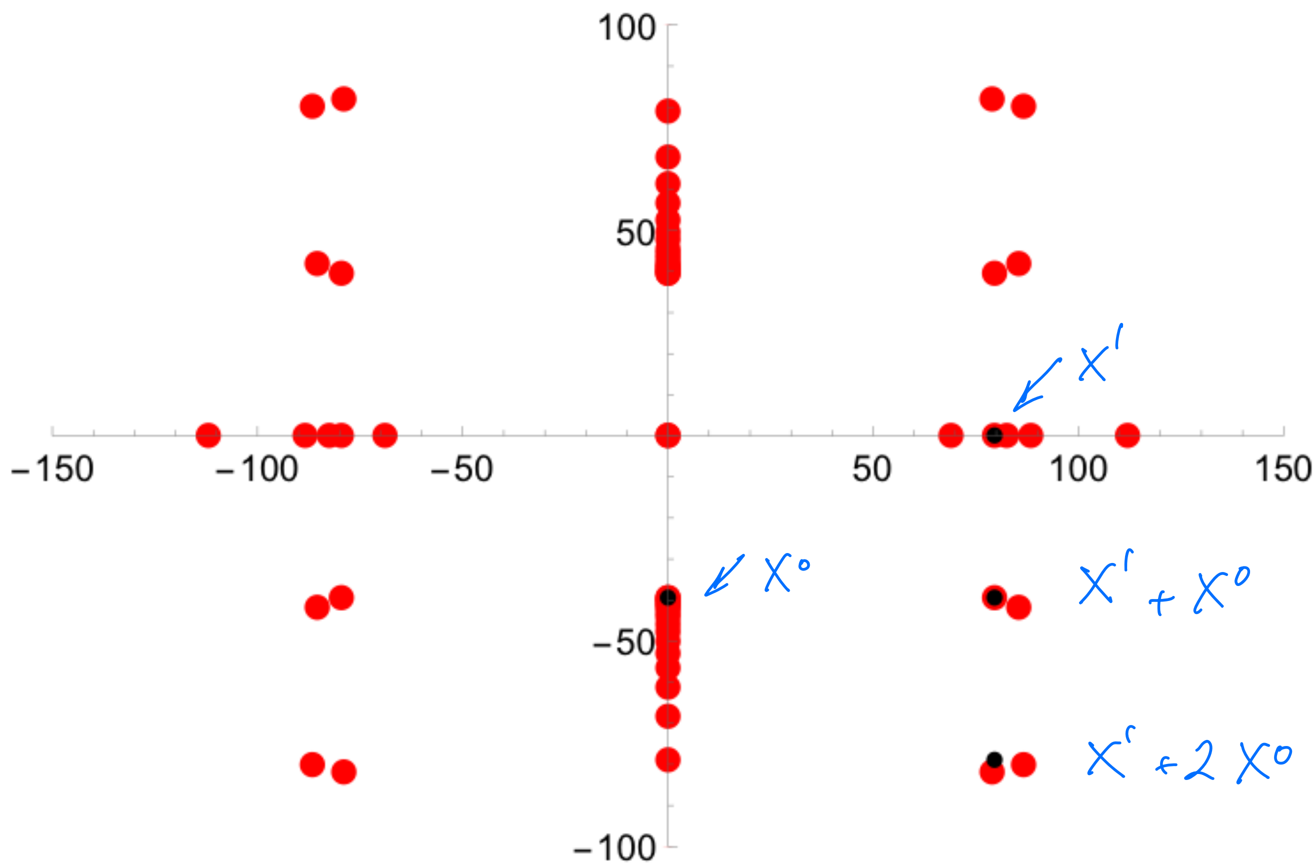
Mapping out the Borel plane

When sufficient number of coefficients a_n (here F_g) are available, leading log singularities in $\hat{\varphi}$ (here $\hat{F}^{(0)}$) visible as accumulation of poles of its Padé approximant

$\uparrow$ approximation by rational f^{cn}

$\Rightarrow$ can identify instanton sectors to which to apply our analysis

Check of method: near MUM pt in MUM frame

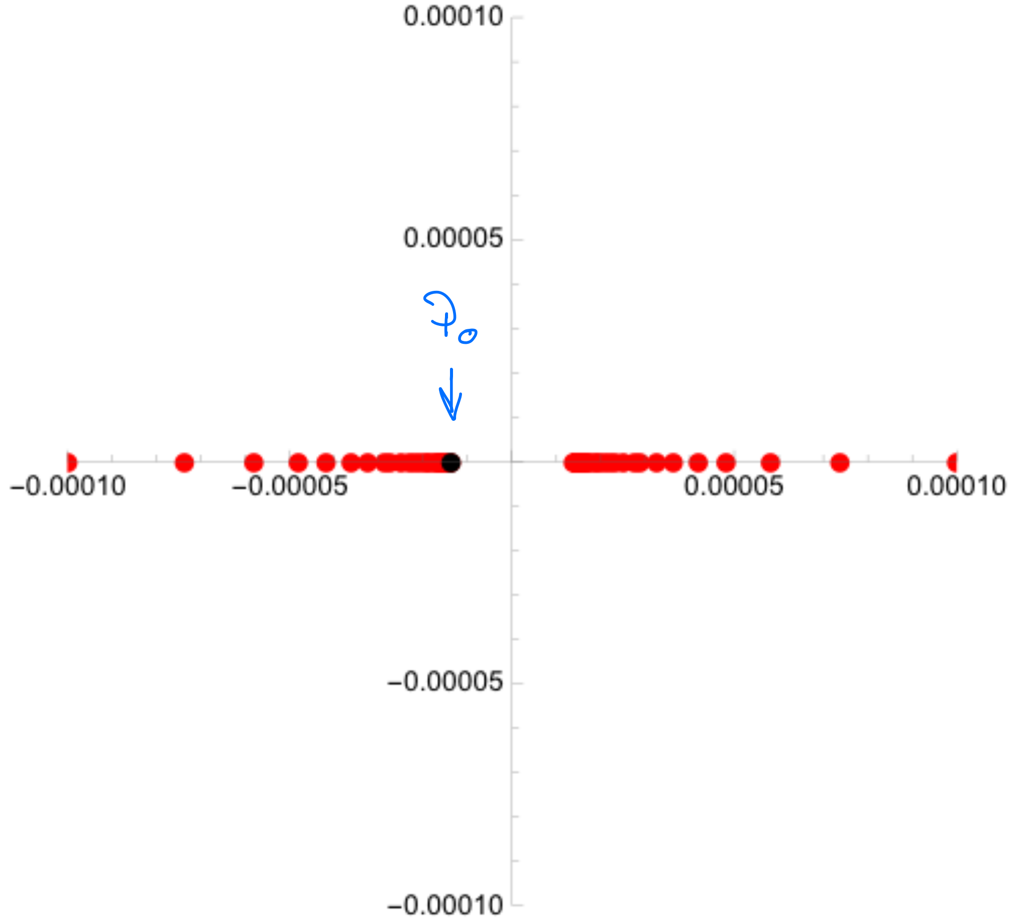


Poles of Padé approximant to $\hat{\mathcal{F}}^{(0)}$, evaluated to $g = 64$

at $z = 10^{-2}$, for the quintic Calabi-Yau.

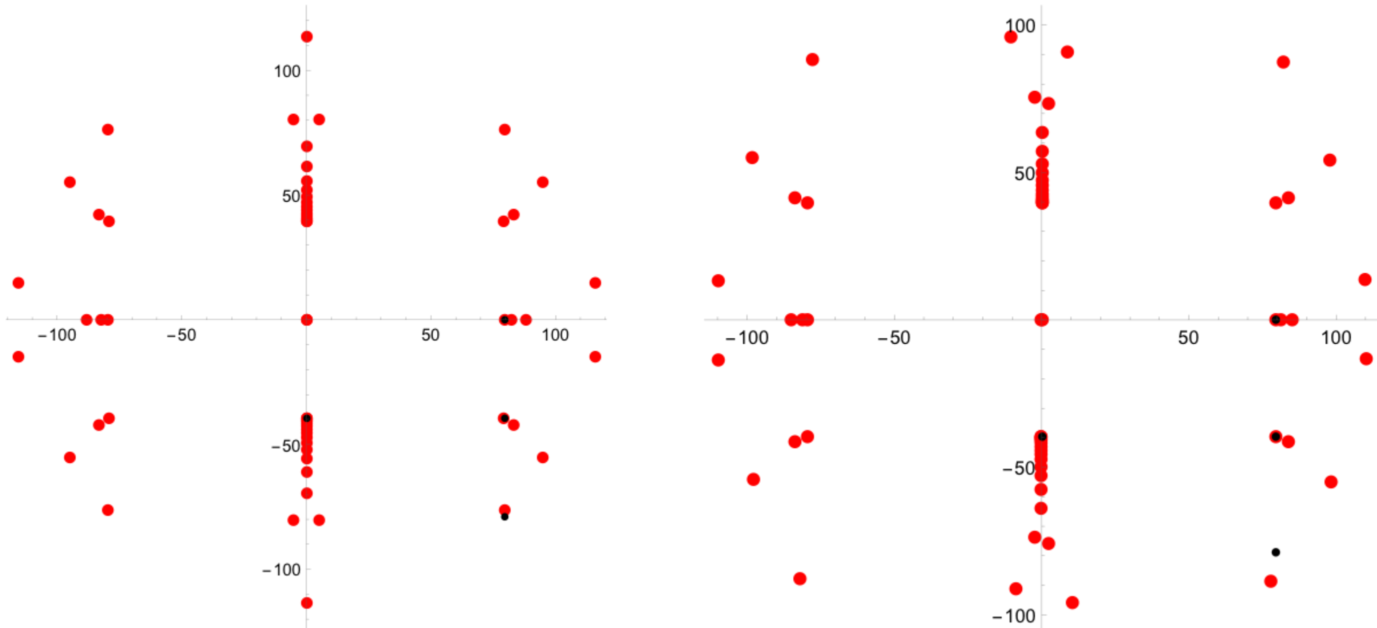
μ ↑ cfld point

Check of method: near cfld in cfld frame



Poles of Padé approximant to $\hat{\mathcal{F}}^{(0)}$, evaluated to $g=64$ at $z=(1-10^{-6})\mu$ for the quintic Calabi-Yau.

Mapping out the Borel plane near the MUM pt

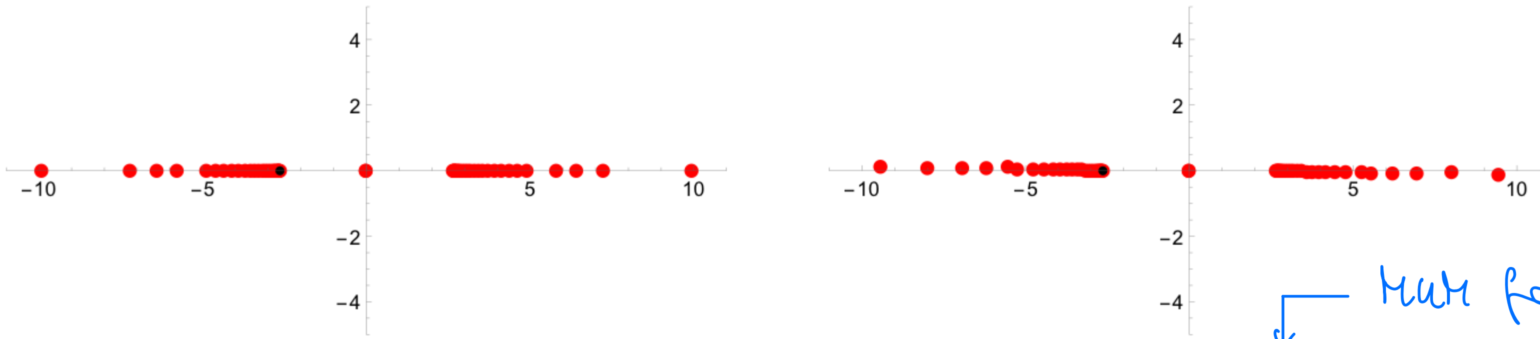


cfld frame

Figure 6: The location of the poles of the Padé approximant to $\widehat{\mathcal{F}}^{(0)}(X^1, P_0)$, on the left, and $\widehat{\mathcal{F}}^{(0)}(P_0, P_1)$, on the right, evaluated to order $g = 64$ at $z = 10^{-2}\mu$. The black dots correspond to the position of the periods $\aleph(mX^0 + nX^1)$, $(m, n) = (1, 0)$ (on the imaginary axis), $(0, 1), (1, 1), (2, 1), \ldots$.

random frame

Mapping out the Borel plane near cfld pt



MUM frame

Figure 8: The location of the poles of the Padé approximant to $\widehat{\mathcal{F}}^{(0)}(X^0, X^1)$, on the left, and $\widehat{\mathcal{F}}^{(0)}(P_0, P_1)$, on the right, evaluated to order $g = 64$ at $z = \frac{5}{6}\mu$. The black dots correspond to the position of the period $\aleph P_0$.

Checking theoretical one instanton amplitudes against asymptotics

Recall leading asymptotics:

$$\mathcal{F}_g \sim \frac{\mathsf{S}_{\mathcal{A}}}{2\pi} \frac{\Gamma(2g-1)}{\mathcal{A}^{2g-1}} \mathcal{F}_0^{(\mathcal{A})} + \frac{\mathsf{S}_{-\mathcal{A}}}{2\pi} \frac{\Gamma(2g-1)}{(-\mathcal{A})^{2g-1}} \mathcal{F}_0^{(-\mathcal{A})} .$$

Extract $\mathcal{F}_0^{(\mathcal{A})}$ via

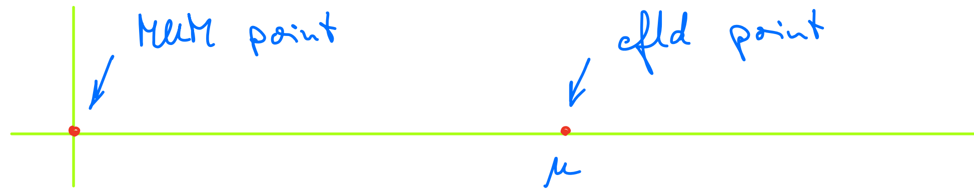
$$s^0_{\mathcal{A},g} = \frac{\mathcal{A}^{2g-1}}{\Gamma(2g-1)} \mathcal{F}_g \quad \xrightarrow{g \gg 1} \quad \frac{\mathsf{S}_{\mathcal{A}}}{\pi} \mathcal{F}_0^{(\mathcal{A})}$$

Checking theoretical one instanton correction against asymptotics

z	Asymptotic estimate I	Asymptotic estimate II	Prediction
$\mu/8$	$1. \times 10^{9}$	-2.0685×10^{-8}	-2.0618×10^{-8}
$\mu/7$	$10000.$	-4.659137×10^{-8}	-4.658992×10^{-8}
$\mu/6$	0.01	$-1.163074023 \times 10^{-7}$	$-1.163074007 \times 10^{-7}$
$\mu/5$	-3.335×10^{-7}	$-3.313309985143 \times 10^{-7}$	$-3.313309985104 \times 10^{-7}$
$\mu/3$	$-5.1510310321251 \times 10^{-6}$	$-5.151031032069825626 \times 10^{-6}$	$-5.151031032069825187 \times 10^{-6}$
$\mu/2$	$-0.000038081205262381317350$	$-0.000038081205262381317350$	$-0.000038081205262381316984$
$5\mu/6$	$-0.00037400000169482575516O$	$-0.00037400000169482575516O$	$-0.000374000001694825754743$
$23\mu/24$	$-0.000499758518253955156739 6$	$-0.000499758518253955156739 6$	$-0.000499758518253955156695 4$

Table 10: Comparison, in the frame (X^0, X^1), between the asymptotic estimate for the normalized genus 0 one-instanton amplitude $\frac{\mathsf{S}_\mu}{\pi}\mathcal{F}_0^{(\mu)}$ and the prediction, using $\mathsf{S}_\mu = 1$, for the example of the quintic.

Checking theoretical one instanton correction against asymptotics



z	Asymptotic estimate I	Asymptotic estimate II	Prediction
$\mu/8$	$1. \times 10^{9}$	-2.0685×10^{-8}	-2.0618×10^{-8}
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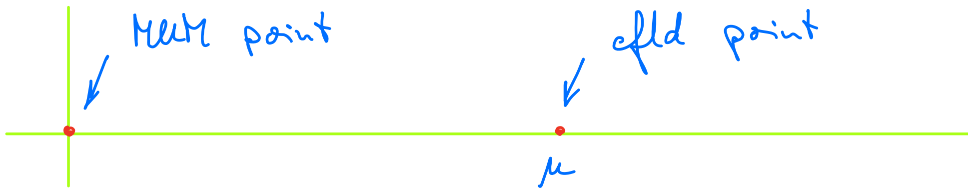
$\mu/3$	$-5.1510310321251 \times 10^{-6}$	$-5.15103103206982562 6 \times 10^{-6}$	$-5.1510310320698251 87 \times 10^{-6}$
$\mu/2$	$-0.0000380812052623813 17350$	$-0.000038081205262381317350$	$-0.000038081205262381316984$
$5\mu/6$	$-0.000374000001694825755160$	$-0.000374000001694825755160$	$-0.000374000001694825754743$
$23\mu/24$	$-0.000499758518253955 1567396$	$-0.0004997585182539551567396$	$-0.0004997585182539551566954$

Table 10: Comparison, in the frame (X^0, X^1), between the asymptotic estimate for the normalized genus 0 one-instanton amplitude $\frac{\mathsf{S}_\mu}{\pi}\mathcal{F}_0^{(\mu)}$ and the prediction, using $\mathsf{S}_\mu = 1$, for the example of the quintic.

Checking theoretical one instanton correction against asymptotics

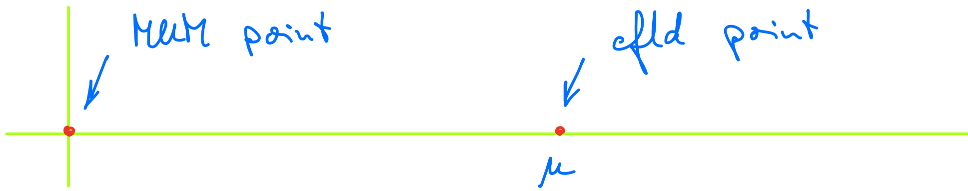


$\mathcal{F}_0^{(0)}$ evaluated in MUM frame for $A \propto P_0$

z	Asymptotic estimate I	Asymptotic estimate II	Prediction
$\mu/8$	$1. \times 10^9$	-2.0685×10^{-8}	-2.0618×10^{-8}
$\mu/7$	$10000.$	-4.659137×10^{-8}	-4.658992×10^{-8}
$\mu/6$	0.01	$-1.163074023 \times 10^{-7}$	$-1.163074007 \times 10^{-7}$
$\mu/5$	-3.335×10^{-7}	$-3.313309985143 \times 10^{-7}$	$-3.313309985104 \times 10^{-7}$
$\mu/3$	$-5.1510310321251 \times 10^{-6}$	$-5.15103103206982 5626 \times 10^{-6}$	$-5.151031032069825187 \times 10^{-6}$
$\mu/2$	$-0.000038081205262381317350$	$-0.000038081205262381317350$	$-0.000038081205262381316984$
$5\mu/6$	$-0.00037400000169482575516 0$	$-0.000374000001694825755160$	$-0.000374000001694825754743$
$23\mu/24$	$-0.000499758518253955156 7396$	$-0.0004997585182539551567396$	$-0.0004997585182539551566954$

Table 10: Comparison, in the frame (X^0, X^1), between the asymptotic estimate for the normalized genus 0 one-instanton amplitude $\frac{\mathsf{S}_\mu}{\pi}\mathcal{F}_0^{(\mu)}$ and the prediction, using $\mathsf{S}_\mu = 1$, for the example of the quintic.

Checking theoretical one instanton correction against asymptotics



$\mathcal{F}_0^{(0)}$ evaluated in MUM frame for $\mathcal{A} \propto P_0$

z	Asymptotic estimate I	Asymptotic estimate II	Prediction
$\mu/8$	$1. \times 10^{9}$	-2.0685×10^{-8}	-2.0618×10^{-8}
$\mu/7$	$10000.$	-4.659137×10^{-8}	-4.658992×10^{-8}
$\mu/6$	0.01	$-1.163074023 \times 10^{-7}$	$-1.163074007 \times 10^{-7}$
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$\mu/2$	$-0.0000380812052623813 17350$	$-0.000038081205262381317350$	$-0.000038081205262381316984$
$5\mu/6$	$-0.000374000001694825755160$	$-0.000374000001694825755160$	$-0.000374000001694825754743$
$23\mu/24$	$-0.000499758518253955156 7396$	$-0.000499758518253955156 7396$	$-0.000499758518253955156 6954$

Table 10: Comparison, in the frame (X^0, X^1), between the asymptotic estimate for the normalized genus 0 one-instanton amplitude $\frac{\mathsf{S}_\mu}{\pi}\mathcal{F}_0^{(\mu)}$ and the prediction, using $\mathsf{S}_\mu = 1$, for the example of the quintic.

contribution from $\mathcal{A} \sim X^0$ instanton sector subtracted

Conclusions

1. Instanton coefficients captured by holomorphic anomaly equations.
2. Integer shift – in units of g_s – of moduli features in exact solution for trans-series correction
3. Integral structure
   - singularities in Borel plane at integral periods
   - Stokes constants topological / enumerative invariants

Conclusions

1. Instanton coefficients captured by holomorphic anomaly equations.
   Why?

2. Integer shift – in units of g_s – of moduli features in exact solution for trans-series connection
   Quantization of moduli space?

3. Integral structure

   - singularities in Borel plane at integral periods
     D-branes?

   - Stokes constants topological / enumerative invariants